Last time : matrix multiplication

$$A \in \mathbb{R}^{d \times m}, \quad B \in \mathbb{R}^{m \times n} \rightsquigarrow AB \in \mathbb{R}^{d \times n}$$

Four ways to calculate :

- $$(AB)_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} = a_{i1} b_{1j} + \dots + a_{im} b_{mj}$$

- multiply i-th row of A by j-th column of B to get (i,j) entry of AB

$$\begin{pmatrix} \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{im} \\ \vdots & \vdots & & \vdots \end{pmatrix} \begin{pmatrix} \cdots & b_{1j} & \cdots \\ \cdots & b_{2j} & \cdots \\ & \vdots & \\ \cdots & b_{mj} & \cdots \end{pmatrix} = \begin{pmatrix} & \vdots & \\ \cdots & a_{i1}b_{1j} + \dots + a_{im}b_{mj} & \cdots \\ & \vdots & \end{pmatrix}$$

- $B = (B_1 \dots B_n) \rightsquigarrow AB = (M_1 \dots M_n)$

where $AB_1 = M_1 \quad \dots \quad AB_n = M_n$

- composition of linear functions

$$f : \mathbb{R}^m \longrightarrow \mathbb{R}^d, \quad f(x) = Ax$$

where $M = AB$

$$g : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad , g(x) = Bx$$

$$f \circ g : \mathbb{R}^n \longrightarrow \mathbb{R}^d \quad , (f \circ g)(x) = Mx$$

The identity matrix $I_n = \begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ 0 & & & & & 1 \end{pmatrix} \in \mathbb{R}^{n \times n}$ is very important, as

$$I_m A = A I_n = A \quad , \forall A \in \mathbb{R}^{m \times n}$$

Note: as opposed from usual algebra with numbers, matrix multiplication is not commutative

$AB \neq BA$, e.g.

$$\begin{pmatrix} 3 & -1 \\ 7 & 6 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 8 & 13 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 7 & 6 \end{pmatrix} = \begin{pmatrix} 13 & 4 \\ 4 & 7 \end{pmatrix}$$

and you can't always divide by matrices

$AB = AC$
$BA = CA$ do NOT imply $B = C$, e.g.

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 7 & -3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 11 & 2 \\ 11 & 2 \end{pmatrix}$$

!!

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 6 & -4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 11 & 2 \\ 11 & 2 \end{pmatrix}$$

(in fancier terms: not all matrices are invertible)

New topic: matrix-matrix equations

$$\underbrace{A}_{\substack{m\times n\\ \text{given}}}\ \underbrace{X}_{\substack{n\times d\\ \text{unknown}}} = \underbrace{B}_{\substack{m\times d\\ \text{given}}}$$

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & -3 & 0 \end{pmatrix}}_{A} \begin{array}{c} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \end{pmatrix} \\ \begin{array}{cc} \underbrace{\phantom{x_{11}}}_{x_1} & \underbrace{\phantom{x_{12}}}_{x_2} \end{array} \end{array} = \begin{array}{c} \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{array}{cc} \underbrace{}_{b_1} & \underbrace{}_{b_2} \end{array} \end{array}$$

$$\Updownarrow$$

$$Ax_1 = b_1 \quad \& \quad Ax_2 = b_2$$

So matrix-matrix equations boil down to a bunch of simultaneous matrix-vector equations

However, there is a more direct and efficient way to solve matrix-matrix equations, namely Gaussian elimination with really big augmented matrices

$$(A\,|\,B) = \left(\begin{array}{cccc|cc} \boxed{1} & 2 & 3 & 4 & 1 & -1 \\ 0 & \boxed{1} & 0 & 1 & 1 & 1 \\ 0 & -1 & -3 & 0 & 1 & 0 \end{array}\right)$$

$$\underset{R3+R2}{\overset{R1-2\cdot R2}{=}} \left(\begin{array}{cccc|cc} \boxed{1} & 0 & 3 & 2 & -1 & -3 \\ 0 & \boxed{1} & 0 & 1 & 1 & 1 \\ 0 & 0 & \boxed{-3} & 1 & 2 & 1 \end{array}\right)$$

$$\overset{R3\cdot\left(-\frac{1}{3}\right)}{=} \left(\begin{array}{cccc|cc} \boxed{1} & 0 & 3 & 2 & -1 & -3 \\ 0 & \boxed{1} & 0 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{array}\right)$$

$$\overset{R1-3R3}{=} \left(\begin{array}{cccc|cc} \boxed{1} & 0 & 0 & 3 & 1 & -2 \\ 0 & \boxed{1} & 0 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{array}\right)$$

pivot variables

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \end{pmatrix} = \begin{pmatrix} x_{11}+3x_{41} & x_{12}+3x_{42} \\ x_{21}+x_{41} & x_{22}+x_{42} \\ x_{31}-\frac{x_{41}}{3} & x_{32}-\frac{x_{42}}{3} \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \\ -\frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

$X_{11} = 1-3s \qquad X_{12} = -2-3t$

free variables

$$x_{21} = 1 - s \qquad x_{22} = 1 - t$$

$$x_{31} = -\frac{2}{3} + \frac{s}{3} \qquad x_{32} = -\frac{1}{3} + \frac{t}{3}$$

$$x_{41} = s \qquad x_{42} = t$$

$$\text{Solution set} = \left\{ \begin{pmatrix} 1-3s & -2-3t \\ 1-s & 1-t \\ -\frac{2}{3}+\frac{s}{3} & -\frac{1}{3}+\frac{t}{3} \\ s & t \end{pmatrix}, t, s \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 1 & -2 \\ 1 & 1 \\ -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 \end{pmatrix} + s \begin{pmatrix} -3 & 0 \\ -1 & 0 \\ \frac{1}{3} & 0 \\ 1 & 0 \end{pmatrix} + t \begin{pmatrix} 0 & -3 \\ 0 & -1 \\ 0 & 1/3 \\ 0 & 1 \end{pmatrix}, t, s \in \mathbb{R} \right\}$$

parametric form of the solution

New topic: diagonal / triangular matrices

DEF 8.1 a $m \times n$ matrix is **square** if $m = n$

DEF 8.2 a square matrix is **diagonal** if its off-diagonal coefficients (a_{ij} for $i \neq j$) are all 0

"main"

$$\begin{pmatrix} 7 & 0 & 0 & 0 \\ \ldots \end{pmatrix}$$

main diagonal

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

DEF 8.3: a square matrix is upper / lower triangular if all entries below / above diagonal (a_{ij} for $i>j$) / (a_{ij} for $i<j$) are 0

$$\begin{pmatrix} 7 & 8 & 6 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 8 \end{pmatrix} \qquad \begin{pmatrix} 7 & 0 & 0 & 0 \\ 5 & -1 & 0 & 0 \\ 2 & 9 & 0 & 0 \\ -3 & 0 & 6 & 8 \end{pmatrix}$$

("strictly" triangular = triangular & diagonal is all 0)

Prop: if $A = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \; a_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 & & 0 \\ & b_2 & \\ 0 & & \ddots \; b_n \end{pmatrix}$

then $AB = \begin{pmatrix} a_1b_1 & & 0 \\ & a_2b_2 & \\ 0 & & \ddots \; a_nb_n \end{pmatrix}$

Ex: $\begin{pmatrix} 2 & x \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 7 & y \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} 14 & 2y-5x \\ 0 & -15 \end{pmatrix}$

Prop: $$\begin{pmatrix} a_1 & & \overset{\text{stuff}}{*} \\ & \ddots & \\ 0 & & a_n \end{pmatrix}\begin{pmatrix} b_1 & & *' \\ & \ddots & \\ 0 & & b_n \end{pmatrix} = \begin{pmatrix} a_1b_1 & & *'' \\ & \ddots & \\ 0 & & a_nb_n \end{pmatrix}$$

$$\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ * & & a_n \end{pmatrix}\begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ *' & & b_n \end{pmatrix} = \begin{pmatrix} a_1b_1 & & 0 \\ & \ddots & \\ *'' & & a_nb_n \end{pmatrix}$$

Ex: A square, its powers are $\underset{A^0}{\underset{\shortparallel}{I_n}}, \underset{A^1}{\underset{\shortparallel}{A}}, A^2, A^3, A^4, \dots\dots$

if $A = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \; a_n \end{pmatrix} \Rightarrow A^k = \begin{pmatrix} a_1^k & & 0 \\ & a_2^k & \\ & & \ddots \\ 0 & & & a_n^k \end{pmatrix}$

(for a general matrix $A = (a_{ij})$, not true that $A^k = (a_{ij}^k)$)

New topic : transposed matrices

DEF 8.4: given a matrix $A = (a_{ij})_{1 \le j \le n}^{1 \le i \le m} \in \mathbb{R}^{m \times n}$

its transpose is $A^T = (a_{ji})_{1 \le j \le n}^{1 \le i \le m} \in \mathbb{R}^{n \times m}$

$$\begin{pmatrix} 2 & 3 & 10 \end{pmatrix} \qquad A^T \begin{pmatrix} 2 & 5 \end{pmatrix}$$

$$A = \begin{pmatrix} 5 & 8 & 12 \end{pmatrix} \qquad A = \begin{pmatrix} 3 & 8 \\ 10 & 12 \end{pmatrix}$$

Note: if a matrix is not square, it changes shape when transposing (rows become columns and vice versa)

$$A = \begin{pmatrix} 1 & a & b \\ x & 2 & c \\ y & z & 3 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & x & y \\ a & 2 & z \\ b & c & 3 \end{pmatrix}$$

<u>Prop</u>:

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(\lambda A)^T = \lambda A^T$
- $(AB)^T = B^T A^T$

Proof: $$(AB)^T_{ij} = (AB)_{ji} = a_{j1}b_{1i} + a_{j2}b_{2i} + \dots$$

$$\shortparallel$$

$$(B^T A^T)_{ij} = B^T_{i1}A^T_{1j} + B^T_{i2}A^T_{2j} + \dots = b_{1i}a_{j1} + b_{2i}a_{j2} + \dots$$

Since all coefficients of $(AB)^T$ and $B^T A^T$ match, they are equal.

Ex: $A = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix}$

$B = \begin{pmatrix} -1 & 5 \\ 6 & 0 \end{pmatrix} \Rightarrow B^T = \begin{pmatrix} -1 & 6 \\ 5 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 16 & 10 \\ 37 & 25 \end{pmatrix} \Rightarrow (AB)^T = \begin{pmatrix} 16 & 37 \\ 10 & 25 \end{pmatrix}$

But $B^T A^T = \begin{pmatrix} 16 & 37 \\ 10 & 25 \end{pmatrix}$, while $A^T B^T = \begin{pmatrix} 23 & 12 \\ 32 & 18 \end{pmatrix}$